A Generalized Theory of Varying Alpha

John D. Barrow, Sean Z.W. Lip
DAMTP, Centre for Mathematical Sciences,
Cambridge University, Wilberforce Road,
Cambridge CB3 0WA, UK

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Abstract

In this paper, we formulate a generalization of the simple Bekenstein-Sandvik-Barrow-Magueijo (BSBM) theory of varying alpha by allowing the coupling constant, ω, for the corresponding scalar field ψ to depend on ψ. We focus on the situation where ω is exponential in ψ and find the late-time behaviours that occur in matter-dominated and dark-energy dominated cosmologies. We also consider the situation when the background expansion scale factor of the universe evolves in proportion to an arbitrary power of the cosmic time. We find the conditions under which the fine structure 'constant' increases with time, as in the BSBM theory, and establish a cosmic no-hair behaviour for accelerating universes. We also find the conditions under which the fine structure 'constant' can decrease with time and compare the whole family of models with astronomical data from quasar absorption spectra.

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I. INTRODUCTION

The electromagnetic fine-structure constant, $\alpha = e^2/\hbar c$, has traditionally been thought of as a fundamental constant of Nature. However, high-redshift observations of quasar absorption spectra have continued to suggest [1, 2, 3, 4, 5] that this 'constant' may exhibit very slow temporal and spatial variation. One theory that generalises Maxwell's theory of electromagnetism and the general theory of relativity in order to accommodate and test these possible variations in α is the BSBM theory of Bekenstein and Sandvik, Barrow and Magueijo [6, 7, 8]. In this theory, space-time variations in α are carried by a scalar field which couples to electrically charged matter; this is an example of what has since become known as a 'chameleonic' scalar field. These variations, in turn, contribute to the space-time geometry and cosmological dynamics. Such a theoretical development is important because almost all published observational bounds on variations in α derive from simply allowing α to become a variable in the conventional equations of physics in which it is strictly a constant. The

1

BSBM theory provides a self-consistent varying-α theory in the way that Brans-Dicke theory [9] describes a varying Newtonian gravitation 'constant', G, via a covariantly conserved Brans-Dicke scalar field $\phi_{BD} \sim G^{-1}$.

In the BSBM theory, the quantities c and $\hbar$ are taken to be constants, and variations in α are ascribed to changes in e, the electron charge. We write $e = e_0 e^\psi$, where $\psi \equiv \psi(x^\mu)$ is a dimensionless scalar field, and e_0 is the present value of e. The BSBM Lagrangian is [7]

$$\mathcal{L} = \sqrt{-g}(\mathcal{L}_g + \mathcal{L}_{mat} + \mathcal{L}_\psi + \mathcal{L}_{em}e^{-2\psi}) \qquad (1)$$

where $\mathcal{L}_g = \frac{R}{16\pi G}$, $\mathcal{L}_\psi = -\frac{\omega}{2}\partial_\mu\psi\partial^\mu\psi$ and $\mathcal{L}_{em} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu}$. Here, R is the Ricci curvature scalar, ω is a coupling constant, and we have defined an auxiliary gauge potential $a_\mu = e^\psi A_\mu$ with corresponding field tensor $f_{\mu\nu} = e^\psi F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, so that the covariant derivative takes its usual form $D_\mu = \partial_\mu + ie_0 a_\mu$. The quantity $\mathcal{L}_{em}$ is usually parametrized by the ratio $\zeta = \mathcal{L}_{em}/\rho$, where ρ is the total baryon energy density. This ratio, ζ, describes the fraction of non-relativistic matter in the universe that contributes to $\mathcal{L}_{em}$. The cosmological value of ζ (denoted ζ_m) has to take into account non-baryonic matter, and thus depends strongly on the nature of dark matter, the nature and constituents of which are currently uncertain. In general, therefore, ζ_m is determined by the relative role of magnetic, B, and electric, E, fields in the dominant cold dark matter content of the universe, and $\zeta_m = (E^2 - B^2)/(E^2 + B^2)$ can take any value between -1 and 1 [7],[8]; also, during the radiation era, $\langle\zeta_m\rangle = 0$. In the situations where $\zeta_m < 0$, a slow logarithmic increase in α occurs during the matter-dominated era but α then tends to a constant when dark energy causes the expansion of the universe to accelerate [8]. Therefore, if α is found to decrease with time during the matter-dominated era, the simple BSBM model with $\zeta_m < 0$ would be inconsistent with this variation.

Observational searches for varying α at cosmological redshifts have used high-precision studies of quasar spectra coupled with many-body calculations of the effects of small changes to α on relativistic corrections to atomic transition frequencies [10]. The most recent analysis by Webb et al [5] of a new set of observations of quasar spectra from the VLT gives the following best fits for $\Delta\alpha/\alpha \equiv (\alpha(z) - \alpha_0)/\alpha_0$, where α_0 denotes the present value of α and z is the redshift of the absorption lines:

$$\Delta\alpha/\alpha = (-0.06 \pm 0.16) \times 10^{-5} \text{ for } z < 1.8,$$
$$\Delta\alpha/\alpha = (0.61 \pm 0.20) \times 10^{-5} \text{ for } z > 1.8.$$

For comparison, an earlier analysis of data from the Keck Telescope gave [1], [2], [4]:

$$\Delta\alpha/\alpha = (-0.54 \pm 0.12) \times 10^{-5} \text{ for } z < 1.8,$$
$$\Delta\alpha/\alpha = (-0.74 \pm 0.17) \times 10^{-5} \text{ for } z > 1.8.$$

All the errors quoted are 1σ errors. Note that, for redshifts $z > 1.8$, the sense in which $\alpha(z)$ varies is different for these two sets of observations. The two samples were obtained from telescopes in different hemispheres of the Earth, which has led to a proposal in [5] that there may be a strong spatial (dipolar) dependence in the variation of α. However, the sense of the overall temporal

variation is unclear, and it is conceivable that α could increase, decrease or remain constant with time during the matter era. Further evidence that α may increase during part of the matter-dominated era can be found from observations based on single ion differential measurements, which place a constraint of $\Delta\alpha/\alpha = (0.566 \pm 0.267) \times 10^{-5}$ at $z = 1.84$ [11], [12] that the authors interpret as an upper limit because of unknown systematics. For a summary of these results, and others, the reader is referred to the recent theoretical reviews by Martins [13] and Uzan [14]. However, note that the observational results announced by Srianand et al [15], which appeared at first to be consistent with no significant redshift variation in α, have been shown to have methodological problems, and a subsequent re-analysis of the same data produced results consistent with the variations found in [1, 2, 3, 4]. For a discussion of some of the observational problems, see [16].

In this paper, we will show how the simple BSBM theory can be generalised in a natural manner by allowing the coupling constant ω in $\mathcal{L}_\psi$ to take on a ψ-dependence, so that $\omega = \omega(\psi)$. We will show that a simple choice of the coupling function, $\omega(\psi)$, permits $\Delta\alpha/\alpha$ to increase or decrease during the matter-dominated era according to the value of a single parameter which vanishes when the new theory reduces to the BSBM case (with constant ω). However, there is still a general pattern to the late-time $\alpha(t)$ behaviour in an expanding universe, and solutions for $\alpha(t)$ generally tend to a particular asymptotic form which is independent of the initial conditions. We shall focus our analysis on the late-time behaviour of the matter-dominated and Λ-dominated eras of a zero-curvature Friedmann universe, but also provide solutions for the more general case in which the scale factor of the universe evolves as $a(t) = t^n$, where n is a constant, which mimics negative curvature dominated expansion when $n = 1$. We find that, as in the original BSBM theory, this extended theory still predicts that any variation in α is rapidly suppressed during the Λ-dominated era, thus allowing complex atoms to persist into the far future (see [17]). Our generalised theory also gives more scope for spatial variations in α to be described because gradients in α can now be driven by gradients in $\omega(\psi)$.

In section II we set up the basic equations of the generalized BSBM theory and explore some of their key properties. In section III we discuss the choice of $\omega(\psi)$ and in section IV we study the cosmological solutions in a dust-dominated flat Friedmann universe. In section V we compare this with the behaviour in a wider range of Friedmann background metrics. In section VI we investigate what happens to the variation of α when the universe undergoes accelerated expansion, dominated by dark energy in the form of a positive cosmological constant. In section VII we summarise these results and compare them with the cosmological data. Finally, section VIII contains some concluding discussion.

II. THE UNDERLYING MODEL AND BACKGROUND EQUATIONS

Consider the modified BSBM Lagrangian

$$\mathcal{L} = \sqrt{-g}(\mathcal{L}_g + \mathcal{L}_{\mathrm{mat}} + \mathcal{L}_\psi + \mathcal{L}_{\mathrm{em}}e^{-2\psi}) \tag{2}$$

3

where $\mathcal{L}_g = \frac{R}{16\pi G}$, $\mathcal{L}_\psi = -\frac{\omega(\psi)}{2}\partial_\mu\psi\partial^\mu\psi$ and $\mathcal{L}_{\rm em} = -\frac{1}{4}f_{\mu\nu}f^{\mu\nu}$, where the coupling ω is now taken to be a function of the scalar field ψ. In order to ensure that the energy density of the scalar field, $\rho = \frac{\omega(\psi)}{2}\dot\psi^2$, is non-negative, we will impose the no-ghost condition $\omega(\psi) \geqslant 0$. As in the BSBM theory, the fine structure 'constant' is determined by

$$\alpha = \alpha_0 \exp(2\psi). \tag{3}$$

Varying (2) with respect to ψ gives the equation of motion for the scalar field:

$$0 = \sqrt{-g}\left(\frac{\omega'(\psi)}{2}\partial_\mu\psi\partial^\mu\psi - 2\mathcal{L}_{\rm em}e^{-2\psi} + \omega(\psi)\partial_\mu\partial^\mu\psi + \omega(\psi)\partial^\mu\psi\frac{\partial_\mu\sqrt{-g}}{\sqrt{-g}}\right). \tag{4}$$

We now specialize to the case of a spatially-flat Friedmann universe with coordinates (t, x^i) and metric $g_{\mu\nu} = \mathrm{diag}(-1, a^2(t), a^2(t), a^2(t))$. We will adopt a system of units in which $G = 1$ and $c = 1$. The modified Friedmann equation, obtained by varying (2) with respect to the metric, is:

$$\frac{\dot a^2}{a^2} = \frac{8\pi}{3}\left(\rho_m\left(1 + |\zeta|e^{-2\psi}\right) + \rho_{nb} + \rho_r e^{-2\psi} + \rho_\psi + \rho_\Lambda\right) \tag{5}$$

Here, an overdot indicates a derivative with respect to the comoving proper time t. In this equation, $\rho_\Lambda = \frac{\Lambda}{8\pi}$ is a constant denoting the density of dark energy, $\rho_{nb} \propto a^{-3}$ is the contribution of non-baryonic matter that is assumed to form the cold dark matter, and $\rho_\psi = \frac{\omega(\psi)}{2}\dot\psi^2$. This equation is unchanged from the corresponding equation for BSBM, since any new contributions due to the ψ-dependence of ω will be of the form $\frac{\partial\mathcal{L}}{\partial\psi}\frac{\partial\psi}{\partial g_{\mu\nu}}$, and we have $\frac{\partial\psi}{\partial g_{\mu\nu}} = 0$. Furthermore, (5) can be approximated by a standard Friedmann equation at late times if the conditions

$$|\zeta|e^{-2\psi} \ll 1 \text{ and } \dot\psi^2\omega \ll \rho_c \tag{6}$$

are satisfied then, where ρ_c represents the dominant form of matter in the universe during the epoch being considered. In this paper we will be mainly interested in the cold dark matter and Λ-dominated eras, in which $\rho_c = \rho_m$ and $\rho_c = \rho_\Lambda$ respectively. (Note that the first condition in (6) is automatically satisfied if ψ increases to infinity at late times. However, this is not a necessary requirement: for example, $|\zeta|$ could be small and ψ could tend to a large constant.)

We will call these two conditions the *consistency conditions*, and will use them to check that our late-time solutions are consistent with our assumption that the solution for the expansion scale factor, $a(t)$, is unaffected by the small variations in ψ that we are interested in modelling. In contrast, the evolution of ψ is significantly affected by the cosmological expansion.

The evolution equation (4) for ψ becomes

$$\ddot\psi + 3H\dot\psi + \frac{\omega'(\psi)}{2\omega}\dot\psi^2 = -\frac{2\zeta\rho_m}{\omega}e^{-2\psi} \tag{7}$$

4

and, after substituting $\omega'(\psi)\dot\psi = \dot\omega$, we can rewrite this as:

$$\left(\omega^{\frac{1}{2}}\dot\psi a^3\right)^{\cdot} \equiv \frac{N}{\omega^{1/2}}e^{-2\psi} \tag{8}$$

where $N \equiv -2\zeta\rho_m a^3$ is a constant. In some sense, this equation is just the BSBM equation in disguise. This can be seen by defining a new time coordinate T with $\frac{d}{dT} = \omega^{1/2}\frac{d}{dt}$, so that $T = \int \omega^{-1/2}dt$. (Since $\omega^{-1/2} > 0$, this is a monotonic reparametrisation.) Then (8) becomes

$$\frac{d}{dT}\left(\frac{d\psi}{dT}a^3\right) = Ne^{-2\psi} \tag{9}$$

which is of the same form as the corresponding BSBM equation for ψ, but with a different time parameter, T. However, this observation is of limited use for finding solutions, because we do not know the behaviour of $\psi(t)$ at the outset, and so we do not know the behaviour of $\omega(\psi(t))$.

Inspection of the conservation equation (7) reveals an important feature of the evolution of ψ (and hence α). In the case $N > 0$, if $\dot\psi$ vanishes anywhere, then $\ddot\psi > 0$ at that turning point, so ψ can be monotonic or have a minimum, but it cannot have a maximum. Similarly, in the case $N < 0$, ψ can be monotonic or have a maximum, but it cannot have a minimum. Therefore, in either case, ψ cannot oscillate, although some analyses [18] of the linearised version of (7) have been misled into deducing oscillatory behaviour for α.

At present, $\omega(\psi)$ is an arbitrary function. However, if the observed behaviour of $\alpha(z)$ and $\psi(t)$ is monotonic, it should be possible (in principle) to obtain the form of $\omega(\psi)$ if our observational data are sufficiently accurate to allow us to fit a function for $\alpha(z)$, and hence determine $\psi(t)$. This can be done as follows. First, note that equation (8) can be re-written as a first-order differential equation for $\omega(t)$:

$$\frac{\dot\psi a^3}{2}\dot\omega + \omega(\dot\psi a^3)^{\cdot} = Ne^{-2\psi}$$

which has the general solution

$$\omega(t) = \frac{2N}{(\dot\psi a^3)^2}\int e^{-2\psi}\dot\psi a^3 dt. \tag{10}$$

Given an expression for $\psi(t)$, we can calculate the right-hand side of (10) in terms of t, and then write $t = t(\psi)$ to obtain an expression for $\omega(\psi)$.

For example, in the matter era with $a \propto t^{2/3}$ and $\rho_m \propto t^{-2}$, the general solution (10) allows us to solve explicitly for $\omega(t)$, given a specified behaviour for $\psi(t)$:

$$\omega(t) = \frac{2N}{(\dot\psi t^2)^2}\int e^{-2\psi}\dot\psi t^2 dt.$$

If, for example, $\psi = A\ln t$, where $A > 0$ and $A \neq 1$, then

$$\omega(t) = \frac{Nt^{-2A}}{A(1-A)} + \frac{k}{t^2} = \frac{Ne^{-2\psi}}{A(1-A)} + ke^{-2\psi/A}.$$

5

On the other hand, if $\psi = \ln t$, then

$$\omega(t) = \frac{1}{t^2}\left(2N\ln t + k\right) = e^{-2\psi}(2N\psi + k),$$

where k is an arbitrary constant. It is easy to check that the consistency conditions (6) hold in both cases (which implies that the approximate cosmological solution $a \sim t^{2/3}$ is valid).

III. A CHOICE OF $\omega(\psi)$

The coupling function $\omega(\psi)$ is a free function and its introduction generalizes the BSBM model, in which ω was a constant. This generalization introduces another way in which spatial inhomogeneity can be created in ψ and α. We are interested here in situations where $\omega(\psi)$ remains non-negative and finite for all real values of ψ. We make the choice $\omega(\psi) \propto e^{\mu\psi}$, where μ is a constant to be determined or constrained by observation. Other forms of $\omega(\psi)$ either take negative values (e.g. $\omega(\psi) = \psi$), blow up at the origin (e.g. $\omega(\psi) = \psi^{-2}$) or are more complicated than the simple form above. Thus, in the remainder of this paper, we will restrict ourselves to considering the form

$$\omega(\psi) = \omega_0 e^{\mu\psi}, \tag{11}$$

where ω_0 and μ are constants. Note that the BSBM theory is recovered in the $\mu \to 0$ limit.

IV. SOLUTIONS IN THE MATTER ERA

We need to investigate the cosmological evolution of ψ in the epoch that includes the astronomical observations of quasars and low-redshift information on the possible evolution of α, so we begin our study in the matter-dominated era. We will assume that the universe contains a perfect 'dust' fluid, with zero pressure, and that the background scale factor is of the form $a(t) = a_0 t^{2/3}$. We will need to check that the consistency conditions (6) hold in order to ensure that we can ignore any corrections to the scale factor evolution from the evolution of $\psi(t)$. We will present exact solutions for certain special cases, but for the general behaviour we will use numerical investigations to provide insight into the evolutionary behaviour of α.

From (7), the general evolution equation for $\psi(t)$ in a Friedmann background with expansion scale factor $a(t)$ is

$$\ddot{\psi} + \frac{\mu}{2}\dot{\psi}^2 + 3\frac{\dot{a}}{a}\dot{\psi} = \frac{Ca_0^3 e^{-(2+\mu)\psi}}{a^3} \tag{12}$$

where $\omega = \omega_0 e^{\mu\psi}$ and $C = N/(\omega_0 a_0^3)$. In the matter-dominated era, the scale factor evolves as $a(t) = a_0 t^{2/3}$, so this equation reduces to

$$\ddot{\psi} + \frac{\mu}{2}\dot{\psi}^2 + \frac{2}{t}\dot{\psi} = \frac{Ce^{-(2+\mu)\psi}}{t^2}. \tag{13}$$

The consistency conditions are

$$|\zeta|e^{-2\psi} \ll 1 \text{and } e^{\mu\psi}\dot{\psi}^2 t^2 \ll 1 \qquad \text{as } t \to \infty.$$

The equation (13) can be transformed into an Emden-Fowler equation using the substitution $\psi = \frac{2}{\mu}\ln\left(\theta t^{-1}\right)$:

$$\ddot{\theta} = \frac{\mu C}{2}\theta^{-\frac{4}{\mu}-1}t^{\frac{4}{\mu}}.$$

Note that $\theta(t)$ must be positive, and that $\ddot{\theta}$ always has the same sign. The consistency conditions become $(\theta/t)^{-4/\mu} \ll 1$ and $\left(\dot{\theta} - \theta t^{-1}\right)^2 \ll 1$. The second condition implies that $\theta(t)$ must grow at most as $t\ln t$ as $t \to \infty$ (and that if $\theta(t) \sim t\ln t$ then the constant of proportionality should be very small). In applying these conditions, however, we note that in the standard cosmology the matter era lasts for a limited period of time before giving way to an accelerated expansion with $p \simeq -\rho$. Thus, for example, if $\mu < 0$, then a solution with θ growing asymptotically faster than t would be acceptable, as long as $\theta(t) \leqslant \kappa t$ throughout the matter epoch (where κ is a small constant).

For two special cases, exact solutions can be found:

Case S1: In the case $\mu = -2$, the governing equation is $\ddot{\theta} = -\frac{C\theta}{t^2}$. This equation is equidimensional and has solutions of the form $\theta(t) = t^k$, where k is a constant. If $C = \frac{1}{4}$, then

$$\theta = \beta_1\sqrt{t} + \beta_2\sqrt{t}\ln t$$

where we require that $\beta_2 > 0$, or that $\beta_2 = 0$ and $\beta_1 > 0$. This solution satisfies both consistency conditions. The corresponding solution for ψ is

$$\psi = \frac{1}{2}\ln t - \ln\left(\beta_1 + \beta_2\ln t\right).$$

In this (very) special case, ψ therefore increases logarithmically at late times and the fine structure 'constant' evolves as

$$\alpha \equiv \exp(2\psi) = \frac{t}{\left(\beta_1 + \beta_2\ln t\right)^2} \sim \frac{t}{(\ln t)^2}.$$

On the other hand, if $C < \frac{1}{4}$, then

$$\theta = \beta_1 t^{\frac{1-\sqrt{1-4C}}{2}} + \beta_2 t^{\frac{1+\sqrt{1-4C}}{2}}$$

at late times, where, again, we require that $\beta_2 > 0$, or that $\beta_2 = 0$ and $\beta_1 > 0$. In the former case, we require $C > 0$ for the first consistency condition to hold; in the latter case, there is no condition on C. (As noted above, it is possible to have a consistent solution in the former case when $C < 0$, as long as β_2 is very small.) The corresponding solution for ψ is

$$\psi = \frac{1}{2}\left(1 - \sqrt{1-4C}\right)\ln t - \ln\left(\beta_2 + \beta_1 t^{-\sqrt{1-4C}}\right).$$

At late times, this solution tends to $\psi = \frac{1}{2}\left(1 - \sqrt{1 - 4C}\right)\ln t - \beta_1 t^{-\sqrt{1-4C}}$ and α evolves as

$$\alpha = \frac{t^{1-\sqrt{1-4C}}}{\left(\alpha_2 + \alpha_1 t^{-\sqrt{1-4C}}\right)^2} \sim t^{1-\sqrt{1-4C}}.$$

Note that, in this case, ψ decreases at late times if and only if $C < 0$.

Finally, if $C > \frac{1}{4}$, we obtain

$$\theta = t^{\frac{1}{2}}\left(\beta_1 \cos\left(\frac{\sqrt{4C-1}}{2}\ln t\right) + \beta_2 \sin\left(\frac{\sqrt{4C-1}}{2}\ln t\right)\right).$$

In this case, physical solutions do not exist because $\theta(t)$ takes negative values.

Case S2: In the case $\mu = -4$, the governing equation becomes $\ddot{\theta} = -2Ct^{-1}$ with the exact solution $\theta = -2Ct\ln t + \beta_1 t + \beta_2$, where β_1 and β_2 are arbitrary constants. Since we require $\theta(t)$ to be positive (at least at late times), this means that a solution exists only if $C < 0$. The corresponding solution for ψ is

$$\psi = -\frac{1}{2}\ln\left(-2C\ln t + \beta_1 + \frac{\beta_2}{t}\right),$$

and $\alpha \sim 1/\ln t$ at late times. This solution decreases at late times, and therefore it is necessary to impose the requirement that $-2C\ln t_{m\Lambda} \ll 1$, where $t_{m\Lambda}$ is the time at which the matter era ends.

We now consider general values of μ. We determine asymptotic forms for the solutions and illustrate the behaviour with numerical plots of ψ against $\ln(\ln t)$) for a wide range of initial conditions. Remarkably, there are just two simple asymptotes:

Case 1: $C > 0$: Here, there are two cases depending on the value of μ. If $\mu < -2$, there is a finite-time singularity at which ψ, and thus α, becomes unboundedly large. However, if $\mu > -2$, all solutions tend to a common asymptote, similar to the behaviour found during the matter era in the standard BSBM scenario described in [8] (Fig. 1). This result does not seem to depend on the initial conditions. The rate at which ψ increases along this asymptote can be made as gentle as desired by taking μ sufficiently large (Fig. 2).

We will now obtain an explicit formula for the asymptote by showing that, at late times,

$$\psi \to \frac{1}{2+\mu}\ln\left((2+\mu)C\ln t\right)$$

and so

$$\alpha \to (\ln t)^{2/(2+\mu)}$$

Note that this is consistent with the earlier BSBM result [8], $\psi \to \frac{1}{2}\ln\left(2C\ln t\right)$, which is recovered when $\mu \to 0$.

In order to extract the asymptotic solution in finer detail we will proceed as in [8]. Using the substitution $x = \ln(t/t_1)$, where t_1 is an arbitrary constant, we can transform (13) to the following equation:

$$\psi'' + \psi' + \frac{\mu}{2}\psi'^2 = Ce^{-(2+\mu)\psi} \tag{14}$$

8

where a prime denotes d/dx. We now write

$$\psi = A\ln(Bx) + \sum_{n=1}^{\infty} a_n x^{-n}.$$

Substituting this series into (14) gives

$$Ax^{-1} \quad + \quad \left(\frac{\mu}{2}A^2 - A - a_1\right)x^{-2} + \sum_{n=2}^{\infty}\left((n-\mu A)(n-1)a_{n-1} - na_n\right)x^{-n-1}$$

$$+ \quad \frac{\mu}{2}\left(\sum_{n=1}^{\infty} na_n x^{-n-1}\right)^2 = C(Bx)^{-(2+\mu)A}\exp\left(-(2+\mu)\sum_{n=1}^{\infty} a_n x^{-n}\right).$$

In order for the leading terms (in x^{-1}) to match, we need $-1 = -(2+\mu)A$ and thus $A = \frac{1}{2+\mu}$. We also need $A = CB^{-(2+\mu)A}$ which forces $B = (2+\mu)C$. This leads to

$$\frac{1}{(2+\mu)x} \quad + \quad \left(\frac{\mu}{2}A^2 - A - a_1\right)x^{-2} + \sum_{n=2}^{\infty}\left((n-\mu A)(n-1)a_{n-1} - na_n\right)x^{-n-1}$$

$$+ \quad \frac{\mu}{2}\left(\sum_{n=1}^{\infty} na_n x^{-n-1}\right)^2 = \frac{1}{(2+\mu)x}\left(e^{-(2+\mu)\sum_{n=1}^{\infty} a_n x^{-n}}\right).$$

The successive terms of $(a_n)_{n=1}^{\infty}$ can now be chosen so as to make the left-hand side vanish. This requires

$$a_1 = -\frac{4+\mu}{2(2+\mu)^2}, \quad a_2 = -\frac{1}{4}\frac{(4+\mu)^2}{(2+\mu)^3}, \text{ etc.}$$

which leads to

$$\frac{1}{(2+\mu)x} = \frac{1}{(2+\mu)x}\exp\left(-(2+\mu)\sum_{n=1}^{\infty} a_n x^{-n}\right) \to \frac{1}{(2+\mu)x}$$

as $x \to \infty$. Therefore, we have

$$\psi = \frac{1}{2+\mu}\ln((2+\mu)Cx) + \sum_{n=1}^{\infty} a_n x^{-n}$$

and

$$\alpha = e^{2\psi} = [(2+\mu)C\ln(t/t_1)]^{\frac{2}{2+\mu}}\exp\left(-\frac{4+\mu}{(2+\mu)^2\ln(t/t_1)}\right) \qquad (15)$$

to leading order.

In order for this solution to hold during the matter era, the value of t_1 should be less than the value of t at the beginning of the matter-dominated era.

Case 2: $C < 0$. The analysis again breaks down into two cases. If $\mu > -2$, then at some finite future time, ψ becomes unboundedly negative, driving α to

zero; we ignore this case with a finite-time singularity. However, if $\mu < -2$, the behaviour mimics the $C > 0, \mu > -2$ behaviour, and solutions tend to a common asymptote

$$\psi \to \frac{1}{2 + \mu} \ln\left((2 + \mu)C \ln(t/t_1)\right)$$

where, again, t_1 is an arbitrary constant. The analysis is similar to the above, but now ψ and α both decrease with time (Figs. 3 and 4).

Summary

To summarize, there are three interesting conclusions to be drawn, assuming this form of $\omega(\psi) = \omega_0 \exp(\mu\psi)$:

(a) if α decreases during the late matter era, we must have $\mu < -2$ and $C < 0$. Note that care must be taken in this case to ensure that the consistency conditions for the Friedmann equation are not violated, since $e^{-2\psi} = [(2 + \mu)C \ln t]^{-\frac{2}{2+\mu}}$ is increasing and, if $-4 < \mu < -2$, then $e^{\mu\psi}\dot{\psi}^2 t^2$ is increasing.

(b) if α increases during the late matter era, we must have $\mu > -2$ and $C > 0$. In this case both consistency conditions are automatically satisfied. This includes the original case of the BSBM theory, in which $\mu = 0$.

(c) it is possible to have special behaviour of α in the case $\mu = -2$, and various conditions on C are required depending on whether α is decreasing or increasing.

V. SOLUTIONS WHEN $a(t) = a_0 t^n$

We now consider the evolution of $\alpha(t)$ in situations where the cosmological scale factor exhibits general power-law behaviour $a = a_0 t^n$, where $n > 0$. The equation of motion for the scalar field becomes

$$\ddot{\psi} + \frac{\mu}{2}\dot{\psi}^2 + \frac{3n}{t}\dot{\psi} = \frac{Ce^{-(2+\mu)\psi}}{t^{3n}} \tag{16}$$

where, as before, $\omega = \omega_0 e^{\mu\psi}$ and $C = N/(\omega_0 a_0^3)$. The substitution $x = \ln(t)$ then yields the equation

$$\psi'' + (3n - 1)\psi' + \frac{\mu}{2}\psi'^2 = Ce^{-(2+\mu)\psi+(2-3n)x} \tag{17}$$

for $\psi(x)$. Two separate cases then arise, depending on whether $\mu = -2$ or $\mu \neq -2$.

Case 1: In the case $\mu = -2$, the evolution equation becomes

$$\ddot{\psi} - \dot{\psi}^2 + \frac{3n}{t}\dot{\psi} = \frac{C}{t^{3n}}. \tag{18}$$

Let $y = \dot{\psi}$. Then

$$\dot{y} = \frac{C}{t^{3n}} - \frac{3n}{t}y + y^2.$$

10

This is a Riccati equation, which can be solved by making the substitution $y = -\frac{\dot{u}}{u}$. Then u satisfies

$$\ddot{u} + \frac{3n}{t}\dot{u} + \frac{C}{t^{3n}}u = 0.$$

The solution of this differential equation is

$$u = \beta_1 t^{\frac{1}{2}(1-3n)} J_{\frac{1-3n}{3n-2}}\left(\frac{2\sqrt{C}(t^{1-\frac{3n}{2}})}{2-3n}\right) + \beta_2 t^{\frac{1}{2}(1-3n)} J_{\frac{3n-1}{3n-2}}\left(\frac{2\sqrt{C}(t^{1-\frac{3n}{2}})}{2-3n}\right)$$

where β_1 and β_2 are arbitrary constants, and $J_m(x)$ is a Bessel function of the first kind. There are two types of late-time behaviour. If $n < \frac{2}{3}$ then

$$
\begin{aligned}
u \quad \approx \quad & \beta_1 t^{-\frac{3n}{2}} \cos\left(\frac{2\sqrt{C}t^{1-\frac{3n}{2}}}{2-3n} - \left(\frac{1-3n}{3n-2}\right)\frac{\pi}{2} - \frac{\pi}{4}\right) \\
& + \beta_2 t^{-\frac{3n}{2}} \cos\left(\frac{2\sqrt{C}t^{1-\frac{3n}{2}}}{2-3n} - \left(\frac{3n-1}{3n-2}\right)\frac{\pi}{2} - \frac{\pi}{4}\right)
\end{aligned}
$$

so, at late times, $\psi \to \text{const.} + \frac{3n}{2}\ln t$ and $\alpha \to t^{3n}$. As noted earlier, neither ψ nor α can exhibit oscillatory behaviour because, when $\dot{\psi}$ is zero in (18), $\ddot{\psi}$ can only have one sign (determined by the sign of C). Thus, ψ cannot have both maxima and minima.

If $n > \frac{2}{3}$, then

$$
\begin{aligned}
u \quad \approx \quad & \beta_1 t^{\frac{1-3n}{2}}\left(\frac{\sqrt{C}t^{\frac{2-3n}{2}}}{2-3n}\right)^{\frac{1-3n}{3n-2}} + \beta_2 t^{\frac{1-3n}{2}}\left(\frac{\sqrt{C}t^{\frac{2-3n}{2}}}{2-3n}\right)^{\frac{3n-1}{3n-2}} \\
= \quad & A_1 + A_2 t^{1-3n}
\end{aligned}
$$

where A_1 and A_2 are new arbitrary constants. Therefore, at late times,

$$\psi \to \text{const.} - \ln\left(\alpha + \beta t^{1-3n}\right) \to \text{const.}$$

and $\alpha(t) \to \text{constant}$. Hence, ψ increases as $\ln t$ if $n < \frac{2}{3}$, but tends to a constant if $n > \frac{2}{3}$. The latter case includes the important scenario of a curvature-dominated Friedmann universe with $n = 1$ and is indicative of the asymptotically constant behaviour of ψ and α also to be expected in the dark-energy-dominated situation with $n \to \infty$. We will discuss this scenario in Section VI.

Case 2: In the case $\mu \neq -2$, we can define new variables by $w = e^{-(2+\mu)\psi+(2-3n)x}$ and $z = \psi'(x)$. This leads to the two-dimensional dynamical system

$$
\begin{aligned}
w' &= (2-3n)w - (2+\mu)zw \\
z' &= -\frac{\mu}{2}z^2 + (1-3n)z + Cw.
\end{aligned}
$$

11

which has three fixed points: $(w, z) = (0,0), (0, \frac{2}{\mu}(1-3n))$ and $\left(\frac{(2-3n)(3n\mu+12n-4)}{2C(2+\mu)^2}, \frac{2-3n}{2+\mu} \right)$. We will call these points a, b and c respectively.

Note that the fixed points a and b are asymptotic fixed points, since, by definition, w never actually reaches zero. On the other hand, point c exists only if $\frac{(2-3n)(3n\mu+12n-4)}{C} > 0$. This fixed point corresponds to the exact solution

$$\psi = \frac{1}{2+\mu} \ln \left[\frac{2C(\mu+2)^2}{(2-3n)(3n(\mu+4)-4)} \right] + \frac{2-3n}{\mu+2} \ln t,$$

and this solution reduces to the one found in [19] for the BSBM ($\mu = 0$) case. We now investigate the stability of these points in order to understand the late-time behaviour.

Critical point a. For $(0,0)$, the Jacobian matrix has the eigenvalues $(2-3n)$ and $(1-3n)$. Therefore, if $n \neq \frac{1}{3}$ and $n \neq \frac{2}{3}$, this critical point is hyperbolic. It is a saddle point if $\frac{1}{3} < n < \frac{2}{3}$, an unstable node if $n < \frac{1}{3}$, and a stable node if $n > \frac{2}{3}$. The case $n < \frac{1}{3}$ is not of physical interest because it derives from background expansion with $p > \rho$.

Critical point b. For $\left(0, \frac{2}{\mu}(1-3n) \right)$, the two eigenvalues of the Jacobian are $(3n-1)$ and $\left(\frac{3n\mu+12n-4}{\mu} \right)$, so the trace of the Jacobian is $T = 6n - 1 + \frac{4}{\mu}(3n-1)$ and the determinant is $D = (3n-1)\left(3n + \frac{4}{\mu}(3n-1) \right)$. It is helpful to note that $T^2 - 4D = \left(\frac{4}{\mu}(3n-1) + 1 \right)^2 > 0$ if $\mu \neq 4(1-3n)$. This means that, in general, the fixed point is either a saddle point (if $D < 0$) or a node (if $T^2 > 4D > 0$).

If $n \neq \frac{1}{3}$ and $n \neq \frac{4}{3(\mu+4)}$, this critical point is hyperbolic. Various types of behaviour can occur depending on the values of n and μ:

- $n > \frac{1}{3}$. Then this point is a saddle if $3n < \frac{4-12n}{\mu}$, and an unstable node if $3n > \frac{4-12n}{\mu}$.

- $n < \frac{1}{3}$: Then this point is a saddle if $3n > \frac{4-12n}{\mu}$, and a stable node if $3n < \frac{4-12n}{\mu}$.

Critical point c. In this case, the Jacobian matrix is

$$\begin{pmatrix} 0 & \frac{(3n-2)(3n\mu+12n-4)}{2(2+\mu)C} \\ C & \frac{2-6n-\mu}{2+\mu} \end{pmatrix}$$

so $T = \frac{2-6n-\mu}{2+\mu}$ and $D = \frac{(2-3n)(3n\mu+12n-4)}{2(2+\mu)}$. As before, there are several cases. Note that we do not distinguish here between whether the point is a node or a focus, since our primary interest is whether the critical point is stable or unstable.

- If $n > \frac{2}{3}$ and $\mu < \frac{4}{3n} - 4$, point c is a saddle point.
  If $n > \frac{2}{3}$ and $\frac{4}{3n} - 4 < \mu < -2$, point c is an unstable node/focus.
  If $n > \frac{2}{3}$ and $\mu > -2$, point c is a saddle point.

12

- If $n < \frac{2}{3}$ and $\mu < -2$, point c is a stable node/focus.

  If $n < \frac{2}{3}$ and $-2 < \mu < \frac{4}{3n} - 4$, point c is a saddle point.

  If $n < \frac{2}{3}$ and $\mu > \frac{4}{3n} - 4$, then point c is a stable node/focus if $\mu > 2 - 6n$, and is an unstable node/focus if $\mu < 2 - 6n$. (Note that this means that if $n < \frac{1}{3}$ and $\mu > \frac{4}{3n} - 4$, point c must be a stable node/focus.)

We can use these results to obtain a general result for the case $n > \frac{2}{3}$ by noting that the only stable critical point is at $(w, z) = (0, 0)$. Therefore, assuming that $z = t\dot{\psi}$ does not diverge to infinity, we are likely to obtain the behaviour $\psi \to$ const. at late times in accelerating universes.

VI. SOLUTIONS IN AN ACCELERATING UNIVERSE

The data from the Oklo natural reactor ($z \approx 0.14$) [20], the analysis of meteorites ($z \approx 0.45$) [21] and laboratory experiments [22] provide very tight bounds on the allowed variation of alpha at low redshifts. We will show that our generalization of the BSBM theory is still consistent with negligible variation in α during the accelerating era of the universe, although we do not introduce the added complexity of relating local terrestrial observations of varying constants to their global cosmological evolution (see Barrow and Shaw [23], [24], [25] for a detailed discussion of this problem).

The cases of power-law expansion with $n > 2/3$ discussed in the last section cover the situation of an accelerating universe dominated by a fluid with equation of state $-\rho < p < -\rho/3$. The remaining situation of special interest is the particular case of a Λ-dominated era, with $p = -\rho$, in which the scale factor evolves according to the de Sitter form

$$a = a_0 e^{Mt},$$

where $M \equiv \sqrt{\Lambda/3}$. The evolution equation (12) for ψ becomes

$$\ddot{\psi} + \frac{\mu}{2}\dot{\psi}^2 + 3M\dot{\psi} = \frac{Ce^{-(2+\mu)\psi}}{e^{3Mt}} \tag{19}$$

where $C = N/(\omega_0 a_0^3)$, and the consistency conditions (6) are $e^{-2\psi} \ll 1$ and $\dot{\psi}^2 e^{\mu\psi} \ll 1$.

Numerical investigation shows that there are two types of solution. One type tends to a constant at late times, in general increasing before it does so. The value of ψ along the eventual constant asymptote depends on the initial conditions. A notable feature of Fig. 5 is that the behaviour of ψ during the initial stages of the evolution can be complicated, but all solutions seem to become constant for $t > t^*$, where t^* is some fixed 'cooling-off' time. Therefore, if the matter era lasts past time t^*, we would expect negligible variation in ψ once the cosmological constant begins to dominate.

We might expect this behaviour from (19) if $\dot{\psi}$ is not too large, because then we have, approximately,

$$\ddot{\psi} = -3M\dot{\psi}.$$

13

Since $3M > 0$, $\dot{\psi}$ is driven exponentially quickly to zero, and ψ tends to a constant.

The second case occurs when $\dot{\psi}$ is large, so that the $\frac{\mu}{2}\dot{\psi}^2$ term comes into play. We then have

$$\ddot{\psi} = - \left(\frac{\mu}{2}\dot{\psi} + 3M\right)\dot{\psi}, \tag{20}$$

and there are two equilibria for $\dot{\psi}$: at $\dot{\psi} = 0$ and $\dot{\psi} = -6M/\mu$. Regardless of the sign of μ, the fixed point at $\dot{\psi} = 0$ is stable and the fixed point at $\dot{\psi} = -\frac{6M}{\mu}$ is unstable. Therefore, depending on the initial conditions, there are two generic behaviours for $\dot{\psi}$: it may either tend to $\pm\infty$ exponentially quickly, or tend to zero. Assuming that finite-time blow-up does not occur, $\dot{\psi}$ should tend to zero at late times, so ψ tends to a constant as found above. Otherwise, we obtain a second class of diverging solutions.

We can also perform an analysis similar to the one performed in Section V for the general power-law case, but this time we will work directly with equation (19). We begin by defining the variables $w = e^{-(2+\mu)\psi - 3Mt}$ and $z = \dot{\psi}$. This leads to the autonomous dynamical system

$$\dot{w} = -3Mw - (2+\mu)zw,$$
$$\dot{z} = -\frac{\mu}{2}z^2 - 3Mz + Cw.$$

Again, this system has three fixed points, at $(w, z) = (0,0), (0, -\frac{6M}{\mu})$ and $\left(-\frac{9M^2(4+\mu)}{2C(2+\mu)^2}, -\frac{3M}{2+\mu}\right)$. As before, we will name these points a, b and c respectively. Again, points a and b are asymptotic fixed points, and point c only exists if $(4+\mu)/C < 0$. The analysis of the stability of these fixed points now follows:

Critical point a. At $(0,0)$, the Jacobian has the two eigenvalues $-3M$. Therefore, the origin is always a stable node.

Critical point b. At $\left(0, -\frac{6M}{\mu}\right)$, the Jacobian has eigenvalues $3M\left(1 + \frac{4}{\mu}\right)$ and $3M$. The trace and determinant are therefore given by $T = 6M\left(\frac{2}{\mu} + 1\right)$ and $D = 9M^2\left(\frac{4}{\mu} + 1\right)$ respectively, and we have $T^2 - 4D = 144 > 0$. Hence, if $-4 < \mu < 0$, this critical point is a saddle. Otherwise, if $\mu > 0$ or $\mu < -4$, it is an unstable node.

Critical point c. If this fixed point exists, the corresponding Jacobian matrix has determinant $D = -\frac{9M^2(4+\mu)}{2(2+\mu)}$ and trace $T = -\frac{6M}{2+\mu}$. Therefore, if $-4 < \mu < -2$, this point is an unstable node, and if either $\mu > -2$ or $\mu < -4$, it is a saddle point.

Note that, for any value of M, the origin $(0,0)$ is the only stable node. Therefore, in an accelerating era, any bounded late-time solution will satisfy $\psi \to$ const., and we have a form of the cosmic no-hair theorem for the evolution of ψ and α in the de Sitter background metric.

VII. SUMMARY AND THE RELATION TO OBSERVATIONS

We are now in a position to summarize the results of the previous sections, having determined the late-time behaviors for both power-law and exponential evolution of the scale factor $a(t)$. For convenience, the different outcomes are displayed in Table 4.1.

Value of μ	Scale factor evolution $a(t) \propto t^n$			$a(t) \propto e^{Mt}$
	$\frac{1}{3} < n < \frac{2}{3}$	$n = \frac{2}{3}$	$n > \frac{2}{3}$	
$\mu > 0$	$\alpha \to t^{\frac{2(2-3n)}{2+\mu}}$ if $C > 0$	$\alpha \to [(2+\mu)C\ln(t/t_1)]^{\frac{2}{2+\mu}}$ if $C > 0$	const	const
$\mu = 0$ (BSBM)	$\alpha \to t^{2-3n}$ if $C > 0$	$\alpha \to 2C\ln(t/t_1)$ if $C > 0$	const	const
$-2 < \mu < 0$	$\alpha \to t^{\frac{2(2-3n)}{2+\mu}}$ if $C > 0$	$\alpha \to [(2+\mu)C\ln(t/t_1)]^{\frac{2}{2+\mu}}$ if $C > 0$	const	const
$\mu = -2$	$\alpha \to t^{3n}$ if $C > 0$	various cases, depending on C	const	const
$\mu < -2$	$\alpha \to t^{\frac{2(2-3n)}{2+\mu}}$ if $C < 0$	$\alpha \to [(2+\mu)C\ln(t/t_1)]^{\frac{2}{2+\mu}}$ if $C < 0$	const	const

Table 1: This table lists the late-time behaviours of $\alpha(t)$ for various choices of μ and possible evolutions of the scale factor $a(t)$, of both power-law and exponential forms. In the power-law case with $n > 2/3$, and in the exponential case, the fine structure 'constant' $\alpha(t)$ tends to a constant value at late times. In other cases, $\alpha(t)$ tracks an increasing asymptote if $\mu > -2$, and a decreasing asymptote if $\mu < -2$.

We can now attempt to constrain the parameters of the theory by using observational data [26]. Since most of the observational evidence arises from the late matter-dominated era, we will use the asymptotic behaviour obtained in earlier sections as the theoretical basis for comparison.

In the matter-dominated era, the redshift-time relation

$$\frac{a(t_1)}{a(t_2)} = \frac{1+z_2}{1+z_1} = \left(\frac{t_1}{t_2}\right)^{2/3}$$

holds. From our results in Section IV above, solutions tend to the asymptote

$$\psi \to \frac{1}{2+\mu}\ln\left((2+\mu)C\ln(t/t_1)\right)$$

where t_1 is an arbitrary constant (corresponding to redshift z_1, say). Then, using (3), we can express the relative shift in the value of the fine-structure constant expected between a redshift $z \gtrsim z_\Lambda \sim 0.5$ and the present by

$$\frac{\Delta\alpha}{\alpha} \quad \to \quad \frac{\alpha(z) - \alpha_0}{\alpha_0} = \frac{\alpha(z) - \alpha_\Lambda}{\alpha_\Lambda}$$

$$= \quad \left(\frac{\ln(1+z_1) - \ln(1+z)}{\ln(1+z_1) - \ln(1+z_\Lambda)}\right)^{\frac{2}{2+\mu}} - 1$$

15

where $\alpha_\Lambda = \alpha_0$ (to an excellent approximation) is the value of α at the redshift signifying the onset of Λ-domination and an accelerating universe.

Thus, defining the new parameters $\nu = 2/(2 + \mu)$ and $Z = \ln(1 + z_1)$, we obtain the following approximate form for the evolution of $\Delta\alpha/\alpha$ during the matter era:

$$\frac{\Delta\alpha}{\alpha} \rightarrow \left(\frac{Z - \ln(1 + z)}{Z - \ln(1 + z_\Lambda)}\right)^\nu - 1. \tag{21}$$

The only constraint on Z is that it should correspond to a redshift before dust domination, i.e., $Z \gtrsim 10$. Taking $z_\Lambda = 0.5$, sampling values of Z between 0.1 and 1000, and values of ν between -100 and 100, indicates that the best-fit parameters generally satisfy the approximate relation $\nu \sim 2.5Z$, or $(2 + \mu)\ln(1 + z_1) \sim 0.8$. The associated χ^2 value is approximately 168.75, with 143 degrees of freedom; this indicates a fit that is reasonable but not particularly strong. Formula (21) generalises the analytic approximation for $\Delta\alpha/\alpha$ obtained in the BSBM theory ($\mu = 0$).

In the above, we have assumed a sharp transition between dust domination and de Sitter expansion at $z = z_\Lambda$, and assumed that ψ and α are constant when $z < z_\Lambda$. A more accurate approximation that evolves smoothly between the dust and Λ-dominated eras could be obtained by assuming a background cosmological evolution in (12) where

$$a^3 = \frac{8\pi G\rho_0 a_0^3}{\Lambda} \sinh^2\left(\frac{\sqrt{3\Lambda}}{2}t + \beta\right). \tag{22}$$

This form of $a(t)$ solves the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3}$$

for a universe containing dust and a cosmological constant. In this model, the fractional densities are $\Omega_m = \text{sech}^2\left(\frac{\sqrt{3\Lambda}}{2}t\right)$ and $\Omega_\Lambda = \tanh^2\left(\frac{\sqrt{3\Lambda}}{2}t\right)$ respectively. Using the approximate values $\Omega_{\Lambda 0} = 0.73$ and $\Omega_{m0} = 0.27$ [27] and $H_0 = 74.2\text{kms}^{-1}\text{Mpc}^{-1}$ [28], we obtain $\Lambda = 1.266340 \times 10^{-35}\text{s}^{-2}$ and $8\pi G\rho_0 = 0.81H_0^2 = 4.684 \times 10^{-36}\text{s}^{-2}$. We have data for

$$\frac{\Delta\alpha}{\alpha} = e^{2(\psi - \psi_0)} - 1 \equiv e^{-2\psi_0}f - 1$$

and we are interested in the evolution of $f(z) \equiv e^{2\psi}$.

The evolution equation for $\psi(t)$, (13), can be transformed into an evolution equation for $f(z)$:

$$\left[8\pi G\rho_0(1 + z)^3 + \Lambda\right](1 + z)\left[f'' + \frac{\mu f'^2}{4f} - \frac{f'^2}{f}\right] - \left[4\pi G\rho_0(1 + z)^3 + 2\Lambda\right]f'$$

$$= 48\pi GD\rho_0(1 + z)^2 f^{-\frac{\mu}{2}},$$

16

where $D = -\frac{\zeta}{4\pi G\omega_0}$. This allows us to determine the behaviour of f for various initial conditions $f(0)$ and $f'(0)$. Note that $f(0) = e^{2\psi_0} > 0$, and, since the present-day variation of α is negligible, we have assumed that $f'(0) = 0$. Thus, our system has three arbitrary parameters: the value of $f(0)$, the value of μ, and the value of D. The contour plots in Fig. 6 illustrate the regions of parameter space for which there is a good fit to the observed data. The best chi-squared statistic obtained using this model is approximately 169 (attained when $f(0) < 1$), which still indicates a fit that is not very strong.

It is important to note that these investigations of the time variation of α may not be conclusive because recent observations of α are consistent with a spatial variation, and the latter will need to be accounted for before an accurate estimate of μ (or an accurate test of whether $\mu \neq 0$) can be obtained. If we are interested in modelling possible spatial variations in α at the redshifts of quasar absorption spectra, the BSBM theory would require inhomogeneity in ζ/ω. Since ω is constant in that theory, it would be necessary for ζ to vary in space. This means that the identity of the dominant form of cold dark matter would vary throughout space, or has a strange character that requires spatial variation in its electric and magnetic composition. This is not appealing in the absence of other evidence because the time evolutions of the densities of these different varieties of pressureless matter will be the same even if they have different ζ values, and, for one value of ζ to be replaced by another, some type of decay would need to occur. However, in the generalized theory explored here, inhomogeneity in α can be caused by spatial variation in $\zeta/\omega(\psi)$ via $\psi(\vec{x})$, and this is not unnatural. This situation will be explored in greater detail elsewhere by extending the results of [29] and [30].

VIII. CONCLUSIONS

We have shown that the BSBM theory of varying alpha naturally generalises to cases in which the scalar field coupling $\omega(\psi) > 0$ has a functional dependence on the scalar field ψ. We have explored the features of the specific case where this dependence is exponential, $\omega(\psi) \propto e^{\mu\psi}$. The BSBM solution, which is recovered when $\mu = 0$, may now be viewed as just one of a range of possible solutions with $\mu > -2$ which all exhibit similar asymptotic cosmological behaviour in dust or dark-energy-dominated eras at late times. In all these cases, α increases slowly with time, or (in the dark-energy-dominated cases) increases and then tends to a constant-valued asymptote. In the case of a de Sitter background, this gives an extension of the cosmic no-hair behaviour familiar from studies of general relativistic cosmology with constant α.

We have also found different possible behaviors in the cases $\mu \leq -2$. In these cases, it is possible for ψ (and α) to decrease with time, and the consistency conditions must be checked numerically in order to ensure that our approximate solution to the Friedmann equation is not changed by the growing gravitational effects of decreasing ψ.

Finally, it is important to note the limitations of this analysis. In this paper, we have confined attention to theories where the late-time behaviour of $a(t)$ is

17

unaffected by the varying scalar field. However, if this was not the case, then new possibilities might arise. We have also highlighted the need for an analysis of the effects of spatial variation of ψ and α in the post-recombination era; this work will be presented elsewhere.

ACKNOWLEDGMENTS

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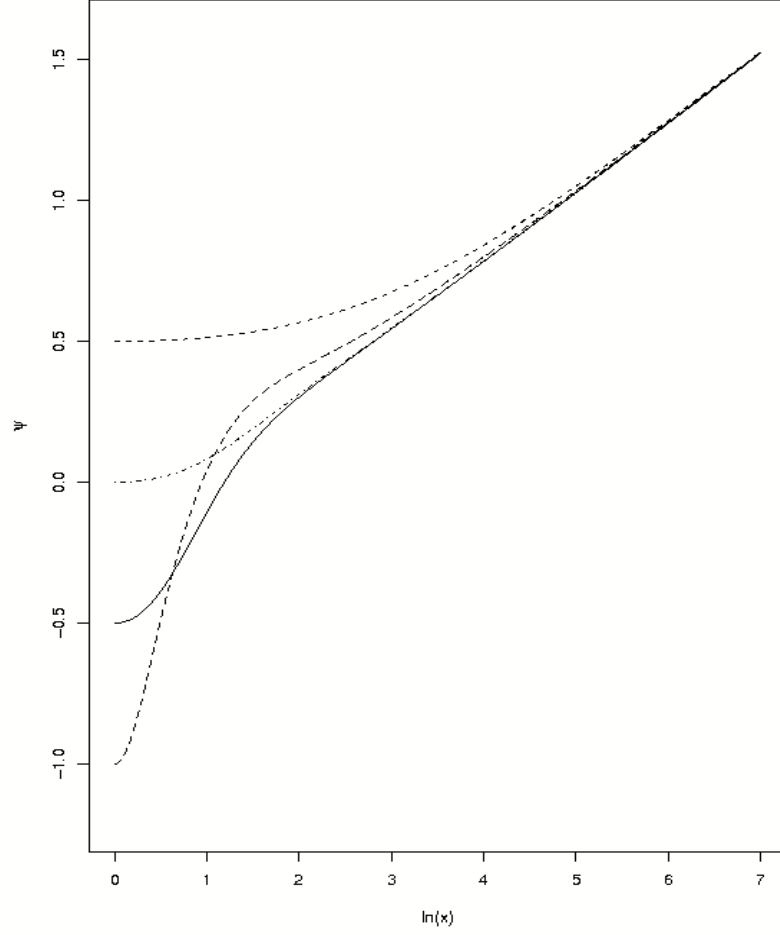[31] J. Magueijo, J.D. Barrow and H.B. Sandvik, Phys. Lett. B **549,** 284 (2002).

Figure 1: A plot of four solutions of equation (13) for the various choices $\psi(\ln x = 0) \in \{-1, -0.5, 0, 0.5\}$ and $d\psi/d(\ln x)|_{\ln x=0} = 0$. Here, x is defined as $\ln(t/t_1)$, where t_1 is an arbitrary constant, and we have chosen (arbitrarily) the parameters $\mu = 2$ and $C = 0.1$. Notice that, in this case ($\mu > -2$), all the solutions eventually converge to a common increasing asymptote, independently of the initial conditions.
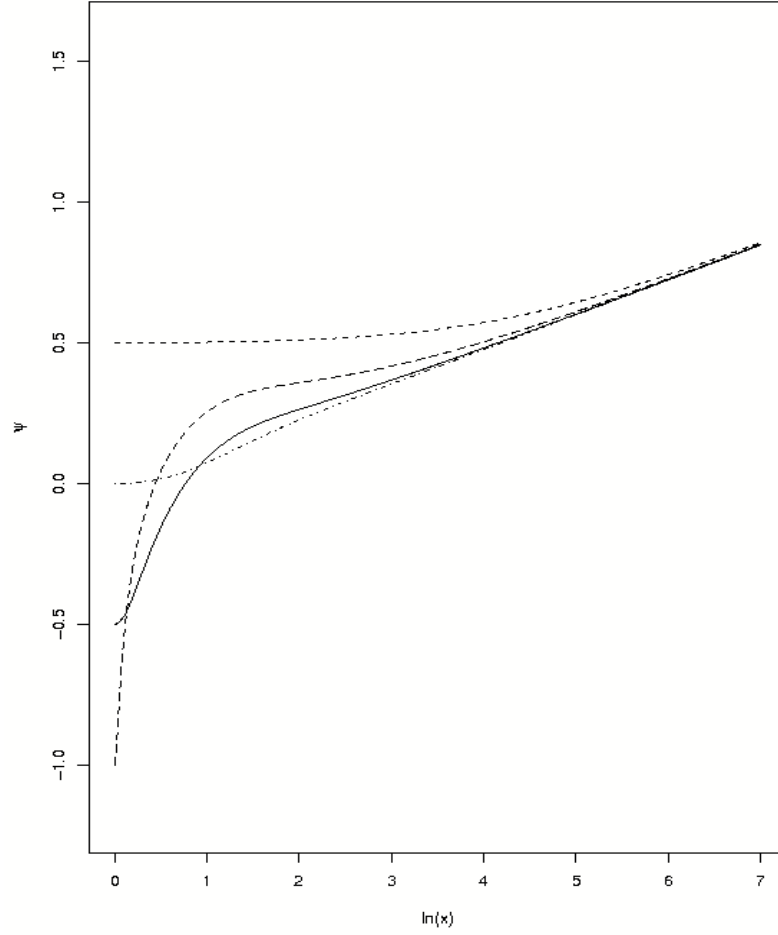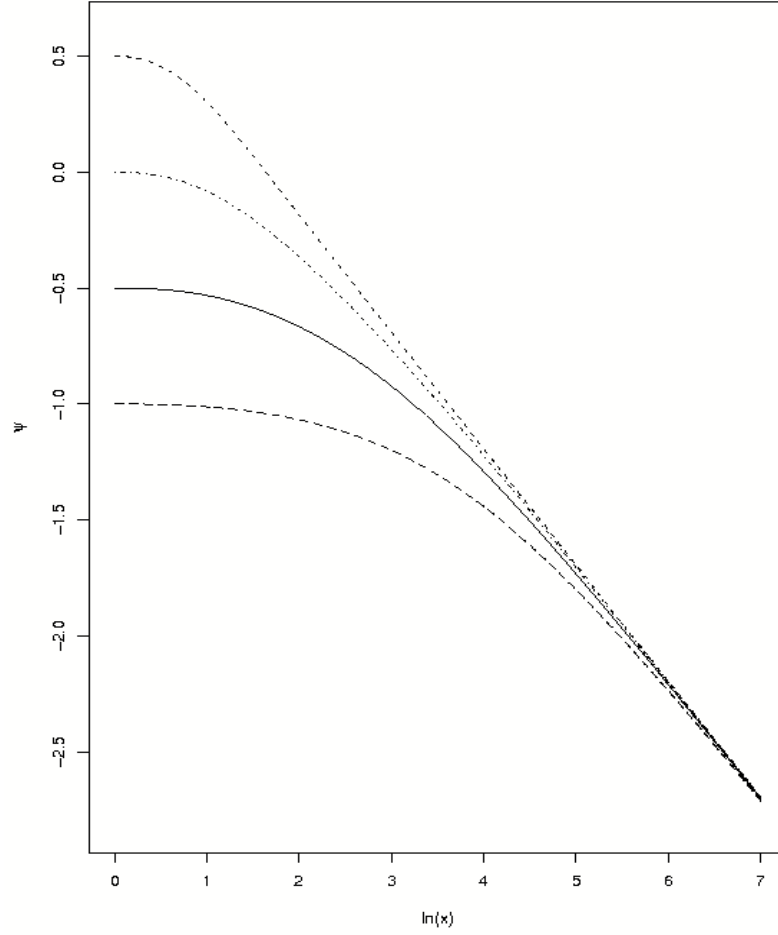
Figure 2: As in Figure 1, but with $\mu = 6$. All solutions still converge to a common increasing asymptote at late times. However, the slope of the asymptote is more gentle than that in Figure 1, because the value of μ has increased. The slope of the asymptote can be made as gentle as desired by taking μ to be sufficiently large.

Figure 3: A plot of four solutions of equation (13) for the various choices $\psi(\ln x = 0) \in \{-1, -0.5, 0, 0.5\}$ and $d\psi/d(\ln x)|_{\ln x=0} = 0$, where, again, x is defined as $\ln(t/t_1)$. The parameter values are now $\mu = -4$ and $C = -0.1$. In this case ($\mu < -2$), the solutions eventually converge to a common decreasing asymptote, and this behaviour is insensitive to the initial conditions.
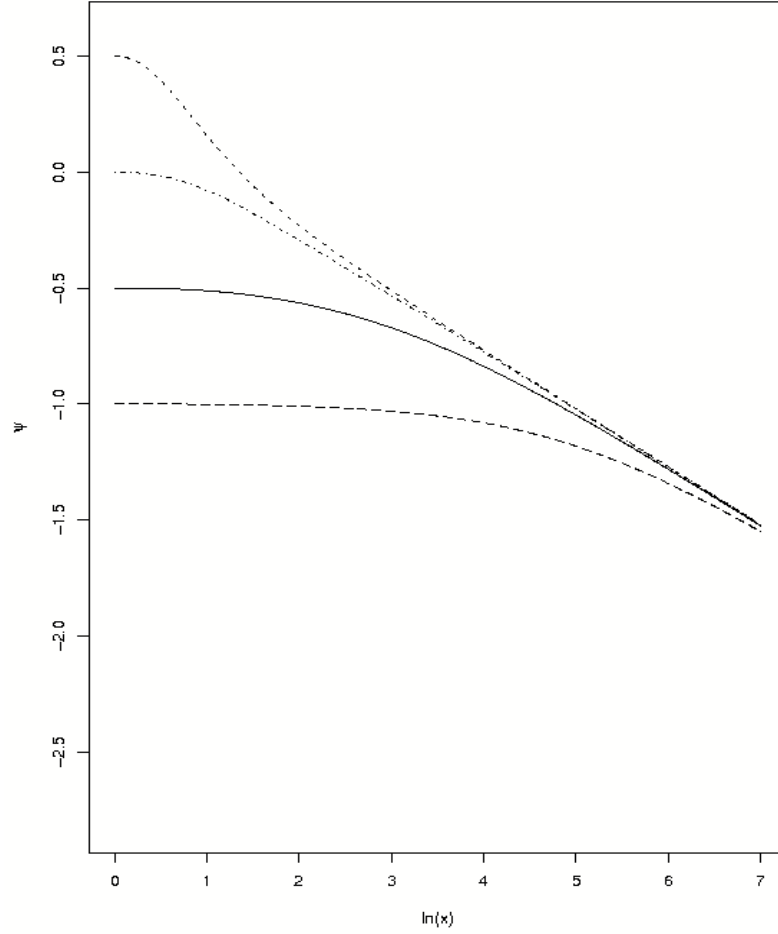
Figure 4: As in Figure 3, but with $\mu = -6$. All solutions still converge to a common decreasing asymptote at late times. However, the slope of the asymptote is more gentle than that in Figure 3 (because μ is now more negative). Again, this slope can be made as gentle as desired by taking a sufficiently large negative μ.
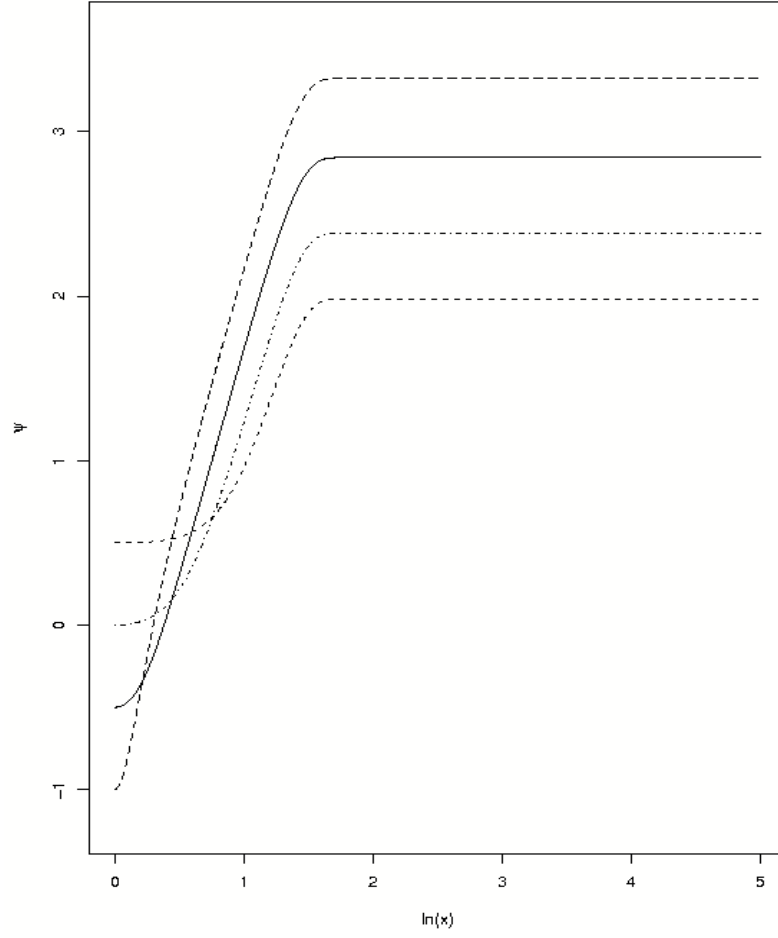
23

Figure 5: As in Figure 1, but these solutions now occur within an accelerating universe. The parameter values are now $\mu = 2$, $C = 0.1$ and $M = 0.01$. Regardless of the initial conditions, the solutions all tend to constant values after a short initial time. This illustrates how the time variation of α is suppressed in a Λ-dominated universe.
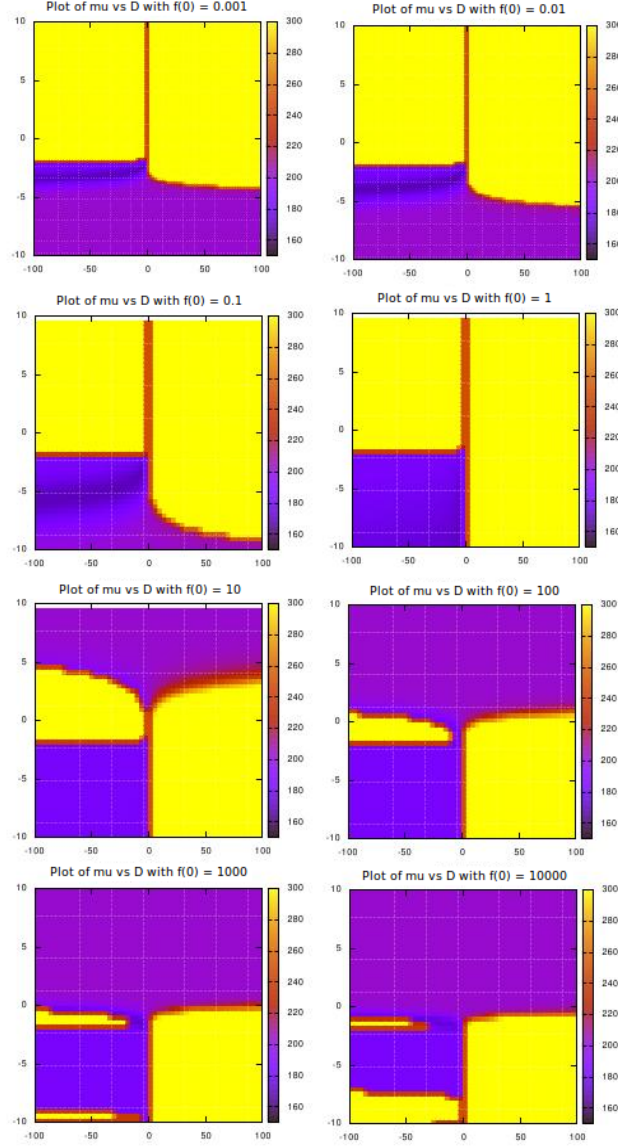
Figure 6: These contour plots display the value of the χ^2 statistic (according to the legend shown in the right-hand color bars) for various choices of D, μ and ψ_0 in a model that exhibits a smooth transition between the dark-matter and dark-energy dominated eras. Each picture corresponds to a different value of $f(0) \equiv e^{2\psi(0)}$ ranging from 10^{-3} to 10^4. The vertical and horizontal axes represent the parameters μ and D, respectively. Darker regions in the plots indicate parameter choices that give rise to a lower χ^2 statistic, and therefore correspond to a better fit.